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ON THE ENDOMORPHISM MONOIDS OF (UNIQUELY) COMPLEMENTED LATTICES

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Abstract. Let L be a lattice with 0 and 1. An endomorphism φ of L is a $\{0,1\}$-*endomorphism*, if it satisfies $0\varphi = 0$ and $1\varphi = 1$. The $\{0,1\}$-endomorphisms of L form a monoid. In 1970, the authors proved that every monoid $\mathcal{M}$ can be represented as the $\{0,1\}$-endomorphism monoid of a suitable lattice L with 0 and 1. In this paper, we prove the stronger result that the lattice L with a given $\{0,1\}$-endomorphism monoid $\mathcal{M}$ can be constructed as a *uniquely complemented lattice*; moreover, if $\mathcal{M}$ is finite, then L can be chosen as a *finite complemented lattice*.

1. Introduction

The endomorphism monoid of a lattice is very special; indeed, the constant maps are always endomorphisms and they are left zeros of the monoid. To exclude them, in [10], the authors considered *bounded* lattices, that is, lattices with zero, 0, and unit, 1, and $\{0,1\}$-endomorphisms, that is, endomorphisms φ satisfying $0\varphi = 0$ and $1\varphi = 1$; and proved the following result:

Theorem 1. *Every monoid $\mathcal{M}$ can be represented as the $\{0,1\}$-endomorphism monoid of a suitable* bounded lattice *L.*

In G. Grätzer and J. Sichler [11], we improved on this result:

Theorem 2. *Every monoid $\mathcal{M}$ can be represented as the $\{0,1\}$-endomorphism monoid of a suitable* complemented lattice *L. Moreover, if $\mathcal{M}$ is finite, then L can be chosen as a* finite complemented lattice.

Theorem 2 is stronger than Theorem 1 in two ways:

(i) we construct a *complemented lattice*;
(ii) for a finite $\mathcal{M}$, we construct a *finite lattice*.

Note that with a few trivial exceptions, the lattice L constructed in [10] is always infinite.

Theorem 2 solves Problem VI.24 of G. Grätzer [8].

The main result of this paper is the construction of a *uniquely complemented lattice* (that is, a lattice in which every element has exactly one complement) to represent a monoid:

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Main Theorem. *Every monoid $\mathcal{M}$ can be represented as the* $\{0,1\}$*-endomorphism monoid of a suitable* uniquely complemented lattice L.

Recall that uniquely complemented lattices are very difficult to construct. R.P. Dilworth [4] solved a long-standing conjecture of lattice theory by proving that not every uniquely complemented lattice is distributive (Boolean). He proved this by examining free lattices with a "free" complement operation; see R.P. Dilworth [5] for a discussion of the background. Free algebras have very special $\{0,1\}$-endomorphism monoids, since every map of the generators can be extended to a $\{0,1\}$-endomorphism. So we were quite surprised that Theorem 2 could be sharpened to the Main Theorem.

Observe that the second clause of Theorem 2 is missing from the Main Theorem; indeed, a finite uniquely complemented lattice is Boolean, and the $\{0,1\}$-endomorphism monoid of a finite Boolean lattice is very special (it is anti-isomorphic to the monoid of all maps of a finite set into itself).

The reader can easily verify that the following version of the Main Theorem holds: *Every full category of algebras is isomorphic to a full subcategory of the category of uniquely complemented lattices and their* $\{0,1\}$*-homomorphisms.* To keep the notation as simple as possible, we only prove the Main Theorem.

The proof of the Main Theorem relies on several results in the literature. In Section 2, we recall a construction—due to M.E. Adams and J. Sichler [1]—of the lattice $\operatorname{Ind}\mathbf{G}$ from a graph $\mathbf{G}$. Section 3 introduces *$\mathcal{C}$-extensions*, a special case of *reduced free products of lattices* of C.C. Chen and G. Grätzer [3] that generalizes some of the techniques originally introduced in R.P. Dilworth [4]. The most important new result in this section is Theorem 12, stating that any sublattice of a reduced free product that contains neither the zero nor the unit is naturally isomorphic to a sublattice of the free product on the same factors; this allows us to apply a result of H. Lakser [13] on simple sublattices in free products to $\mathcal{C}$-extensions.

The construction of the uniquely complemented lattice representing a given monoid is introduced in Section 4. It is based on V. Koubek and J. Sichler [12]. Section 5 proves that this construct has many simple sublattices. Finally, in Section 6, we put all these pieces together to construct the lattice L for the Main Theorem.

To prove Theorem 2, we need a different construction, which is presented in Section 7.

Notation. For general notation, we refer the reader to G. Grätzer [9]. In particular, $\mathfrak{M}_3$ denotes the five-element nondistributive modular lattice.

Let L be a *bounded lattice*, that is a lattice with zero and unit. Then 0_L and 1_L will denote the zero and unit, respectively, unless special notation is introduced. A sublattice S of the bounded lattice L is called *confined*, if $0_L,\ 1_L \notin S$. The $\{0,1\}$-*endomorphism monoid* of L will be denoted by $\operatorname{End}_{\{0,1\}} L$.

For the lattices K and L, we denote by $K * L$ the *free product* of K and L. We view K and L as sublattices of $K * L$. If K and L are bounded, we also have $K *_{\{0,1\}} L$, the $\{0,1\}$-*free product* of K and L.

2. Graphs

In this paper, a *graph* $\mathbf{G} = \langle V, E\rangle$ is a nonempty set V (the *vertices* of $\mathbf{G}$) and a set E of two-element subsets of V (the *edges* of $\mathbf{G}$), that is, $\mathbf{G}$ is an undirected graph with no loops.

For a graph $\mathbf{G}$, let $\operatorname{Ind}\mathbf{G}$ denote the set of all its *finite independent subsets* (that is, finite subsets not containing an edge), partially ordered by set inclusion, with a new unit element, $1_{\mathbf{G}}$. Note that $\{x\} \in \operatorname{Ind}\mathbf{G}$, for any $x \in V$.

The following observations are from V. Koubek and J. Sichler [12].

Lemma 3. *If* $\mathbf{G}$ *is a finite, automorphism-free, connected graph with more than two vertices, then* $\operatorname{Ind}\mathbf{G}$ *is a simple complemented lattice.*

Let $\operatorname{Aut}\mathbf{G}$ denote the automorphism group of $\mathbf{G}$.

Lemma 4. *If* $\operatorname{Ind}\mathbf{G}$ *is a finite simple lattice, then* $\operatorname{End}_{\{0,1\}}\operatorname{Ind}\mathbf{G}$ *is isomorphic to* $\operatorname{Aut}\mathbf{G}$.

3. $\mathcal{C}$-extensions

The verification of the first construction is based on Theorem 16, stated at the end of this section. We need to recall some definitions and results. We are going to state these in simplified settings, sufficient for the applications in this paper. All the results in this section are based on C.C. Chen and G. Grätzer [3], and on G. Grätzer [6] and [7]. Theorem 11 is from these papers (see also G. Grätzer [8], which contains all the necessary results); Theorem 15 is H. Lakser's result, see [13]; Theorems 12 and 16 appear to be new.

Definition 5. Let X be an arbitrary set. The set $\mathbf{P}(X)$ of polynomials in X is the smallest set satisfying (i) and (ii):

(i) $X \subseteq \mathbf{P}(X)$.
(ii) If p, $q \in \mathbf{P}(X)$, then $(p \wedge q)$, $(p \vee q) \in \mathbf{P}(X)$.

Let K be an at most uniquely complemented lattice; let C be the set of elements of K with no complements. Let $C' = \{\, c' \mid c \in C \,\}$ be a copy of C disjoint from K, where the assignment $c \mapsto c'$ is a bijection. We define

$$\mathcal{C} = \{\, \langle c, c' \rangle \mid c \in C \,\} \cup \{\, \langle c', c \rangle \mid c \in C \,\},$$

a binary relation on $K \cup C'$. We are going to construct a $\{0,1\}$-extension of K in which a pair of elements is complementary iff it is complementary in K or the pair is $\mathcal{C}$-related (that is, the pair is of the form $\{c, c'\}$, for some $c \in C$).

Let F denote the bounded free lattice on C'. Let $Q = K \cup F$ and let $\mathbf{P}(Q)$ denote the set of all polynomials over Q.

For a lattice A, we define $A^b = A \cup \{0^b, 1^b\}$, where $0^b, 1^b \notin A$; we order A^b by the rules:

$$0^b < x < 1^b, \quad \text{for all } x \in A.$$
$$x \leq y \text{ in } A^b \quad \text{iff} \quad x \leq y \text{ in } A, \text{ for } x, y \in A.$$

Thus A^b is a bounded lattice. To simplify the discussion, we use the notation $K_0 = K$ and $K_1 = F$.

Definition 6. Let $p \in \mathbf{P}(Q)$ and $i \in \{0,1\}$. The *upper i-cover of p*, in notation, $p^{(i)}$, is an element of K_i^b defined as follows:

(i) For $a \in Q$, we have $a \in K_j$, for exactly one j; if $j = i$, then $a^{(i)} = a$; if $j \neq i$, then $a^{(i)} = 1^b$.

(ii)
$$(p \wedge q)^{(i)} = p^{(i)} \wedge q^{(i)},$$
$$(p \vee q)^{(i)} = p^{(i)} \vee q^{(i)},$$

where $\wedge$ and $\vee$ on the right side of these equations are to be taken in K_i^b.

The definition of the *lower i-cover* of p, in notation, $p_{(i)}$, is analogous, with 0^b replacing 1^b in (i).

Definition 7. For p, $q \in \mathbf{P}(Q)$, set $p \subseteq q$ iff it follows from the following rules:

(C) $\quad p^{(i)} \leq q_{(i)}, \quad$ for some $i \in \{0, 1\}$;

($_\wedge$W) $\quad p = p_0 \wedge p_1, \quad$ where $\quad p_0 \subseteq q \quad$ or $\quad p_1 \subseteq q$;

($_\vee$W) $\quad p = p_0 \vee p_1, \quad$ where $\quad p_0 \subseteq q \quad$ and $p_1 \subseteq q$;

(W$_\wedge$) $\quad q = q_0 \wedge q_1, \quad$ where $\quad p \subseteq q_0$ and $\quad p \subseteq q_1$;

(W$_\vee$) $\quad q = q_0 \vee q_1, \quad$ where $\quad p \subseteq q_0 \quad$ or $\quad p \subseteq q_1$.

It is not difficult to prove that the relation $\subseteq$ is a quasi-ordering, so we can define:

$$\begin{aligned} &p \equiv q \quad \text{iff} \quad p \subseteq q \text{ and } q \subseteq p && (p,\ q \in \mathbf{P}(Q));\\ &R(p) = \{\, q \mid q \in \mathbf{P}(Q) \text{ and } p \equiv q \,\} && (p \in \mathbf{P}(Q));\\ &R(Q) = \{\, R(p) \mid p \in \mathbf{P}(Q) \,\};\\ &R(p) \leq R(q) \quad \text{iff} \quad p \subseteq q. \end{aligned}$$

Lemma 8. *$R(Q)$ is a lattice; in fact,*
$$R(p) \wedge R(q) = R(p \wedge q),$$
$$R(p) \vee R(q) = R(p \vee q).$$
Furthermore, if a, b, c, $d \in K_i$, $i \in \{0, 1\}$, and
$$a \wedge b = c,$$
$$a \vee b = d,$$
in K_i, then
$$R(a) \wedge R(b) = R(c),$$
$$R(a) \vee R(b) = R(d);$$
and if x, $y \in Q, x \neq y$, then $R(x) \neq R(y)$.

By Lemma 8, the assignment
$$a \mapsto R(a), \quad a \in K_i,$$
defines an embedding of K_i into $R(Q)$, for $i \in \{0, 1\}$. Therefore, by identifying $a \in K_i$ with $R(a)$, we get each K_i as a sublattice of $R(Q)$, and hence $Q \subseteq R(Q)$. It is also obvious that the partial ordering induced by $R(Q)$ on Q agrees with the original partial ordering.

Theorem 9. *$R(Q)$ is a free product $K * F$ of K and F.*

Now we shall define a S subset of $\mathbf{P}(Q)$, forcing any ordered pair $\langle c, c' \rangle \in \mathcal{C}$ to become a complemented pair.

Definition 10. For $p \in \mathbf{P}(Q)$, let $p \in S$ be defined by induction on the rank of p:

(i) For $p \in Q$, let $p \in S$ iff $p \in K_i - \{0_{K_i}, 1_{K_i}\}$, for $i = 0$ or $i = 1$.
(ii) For $p = q \wedge r$, let $p \in S$ iff q, $r \in S$ and the following two conditions hold:
  (ii$_1$) $p \subseteq 0_{K_i}$ fails, for $i = 0$ and $i = 1$;
  (ii$_2$) there is no $c \in C$ for which $p \subseteq c \wedge c'$.
(iii) For $p = q \vee r$, let $p \in S$ iff q, $r \in S$ and the following two conditions hold:
  (iii$_1$) $1_{K_i} \subseteq p$ fails, for $i = 0$ and $i = 1$;
  (iii$_2$) there is no $c \in C$ for which $c \vee c' \subseteq p$.

Now we set

$$\widehat{K} = \{0_{\widehat{K}}, 1_{\widehat{K}}\} \cup \{\, R(p) \mid p \in S \,\},$$

and partially order $\widehat{K}$ by

$$\begin{aligned} &0_{\widehat{K}} < R(p) < 1_{\widehat{K}}, \quad \text{for } p \in S,\\ &R(p) \leq R(q) \quad \text{iff} \quad p \subseteq q. \end{aligned}$$

If we identify $a \in K_i - \{0_{K_i}, 1_{K_i}\}$ with $R(a)$, $i \in \{0, 1\}$, then we get the setup we need:

Theorem 11. *$\widehat{K}$ satisfies the following properties:*

(i) *$\widehat{K}$ contains K and F as $\{0, 1\}$-sublattices.*
(ii) *A pair of elements is complementary in $\widehat{K}$ iff they form a complementary pair in K or they are $\mathcal{C}$-related.*

It follows that $\widehat{K}$ is an at most uniquely complemented lattice and every element of K has a complement in $\widehat{K}$.

In the literature, $\widehat{K}$ is called the *$\mathcal{C}$-reduced free product* of K and F. We shall call $\widehat{K}$ the *$\mathcal{C}$-extension* of K.

The following consequence of Theorems 9 and 11 lays the foundation for Theorem 16.

Theorem 12. *Let A be a confined sublattice of $\widehat{K}$, that is, $0_{\widehat{K}}$, $1_{\widehat{K}} \notin A$. Then every element of A is of the form $R(p)$, where $p \in S$. The identity map, $R(p) \mapsto R(p)$, is an embedding of A into $K * F$.*

An alternative description of the $\mathcal{C}$-extension $\widehat{K}$ is the following. Let

$$\overline{K} = K *_{\{0,1\}} F,$$

and let Θ be the smallest congruence on $\overline{K}$ such that

$$\begin{aligned} c \wedge c' &\equiv 0_{\bar{K}} \quad (\Theta),\\ c \vee c' &\equiv 1_{\bar{K}} \quad (\Theta), \end{aligned}$$

for all $c \in C$.

Lemma 13. *$\widehat{K}$ is isomorphic to $\overline{K}/\Theta$.*

We use this in the verification of the extension theorem:

Theorem 14. *Every $\{0, 1\}$-endomorphism φ of K has a unique extension $\widehat{\varphi}$ to a $\{0, 1\}$-endomorphism of the $\mathcal{C}$-extension $\widehat{K}$ of K.*

Proof. Let $\kappa\colon \overline{K} \to \widehat{K}$ denote the surjective $\{0,1\}$-homomorphism with $\operatorname{Ker}\kappa = \Theta$.

For any $c \in K - C$, let c^* denote the unique complement of c in K. Given a $\varphi \in \operatorname{End}_{\{0,1\}} K$, we define a map $\overline{\varphi}$ of the generating set $K \cup C'$ of $\overline{K}$ into itself as follows:

$$x\overline{\varphi} = \begin{cases} x\varphi, & \text{for all } x \in K;\\ (c\varphi)', & \text{if } x = c' \in \overline{K} \text{ and } c\varphi \in C;\\ (c\varphi)^*, & \text{if } x = c' \in \overline{K} \text{ and } c\varphi \in K - C. \end{cases}$$

Then $\overline{\varphi}$ extends φ, and determines a unique $\{0,1\}$-endomorphism of $\overline{K}$, again denoted as $\overline{\varphi}$.

Next we show that $\overline{\varphi} \in \operatorname{End}_{\{0,1\}} \overline{K}$ preserves Θ (introduced before Lemma 13), that is, $u \equiv v$ (Θ) implies that $u\overline{\varphi} \equiv v\overline{\varphi}$ (Θ). Since Θ is the congruence generated by the relations $c \wedge c' \equiv 0_{\bar{K}}$ (Θ) and $c \vee c' \equiv 1_{\bar{K}}$ (Θ), for all $c \in C$, it is sufficient to prove that $(c \wedge c')\overline{\varphi} \equiv 0_{\bar{K}}$ (Θ) and $(c \vee c')\overline{\varphi} \equiv 1_{\bar{K}}$ (Θ), for all $c \in C$.

Let $c \in C$. Then $(c \vee c')\overline{\varphi} = c\overline{\varphi} \vee c'\overline{\varphi} = c\varphi \vee c'\overline{\varphi}$. There are two cases to consider.

Case 1. $c\varphi \in C$. Hence $c'\overline{\varphi} = (c\varphi)'$. But then $(c \vee c')\overline{\varphi} = c\varphi \vee (c\varphi)'$ and, since $1_{\overline{K}}\overline{\varphi} = 1_{\overline{K}}$, the pair $\langle (c \vee c')\overline{\varphi}, 1_{\overline{K}}\overline{\varphi}\rangle$ belongs to Θ.

Case 2. $c\varphi \in K - C$. Hence $c'\overline{\varphi} = (c\varphi)^*$, so that $(c \vee c')\overline{\varphi} = c\varphi \vee (c\varphi)^* = 1_K$ in K. Thus $(c \vee c^*)\overline{\varphi} \equiv 1_K$ (Θ).

Calculating similarly for the meet, we conclude that $\overline{\varphi}$ preserves Θ.

Consequently, the kernel of $\overline{\varphi}\kappa$ contains the kernel Θ of the surjective homomorphism $\kappa\colon \overline{K} \to \widehat{K}$, and hence there exists a unique $\{0,1\}$-endomorphism $\widehat{\varphi}$ of $\widehat{K}$, completing the diagram

$$\begin{array}{ccc} \overline{K} & \xrightarrow{\overline{\varphi}} & \overline{K} \\ {\scriptstyle\kappa}\big\downarrow & & {\scriptstyle\kappa}\big\downarrow \\ \widehat{K} & \xrightarrow{\widehat{\varphi}} & \widehat{K} \end{array}$$

and thus extending the $\{0,1\}$-endomorphism φ of K. The lattice $\widehat{K}$ is at most uniquely complemented and the κ-image of the set $K \cup C'$ generates $\widehat{K}$, and so the extension $\widehat{\varphi}$ of φ is unique. $\square$

Now we state the result of H. Lakser [13].

Theorem 15. *Let S be a finite simple lattice with more than two elements. We assume that S is not isomorphic to $\mathfrak{M}_3$.*

If S is isomorphic to a sublattice S_1 of a free product $A * B$ of the lattices A and B, then S_1 is contained in A or B.

The following result is crucial in the verification of the first construction:

Theorem 16. *Let S be a finite simple lattice with more than two elements. We assume that S is not isomorphic to $\mathfrak{M}_3$. Let K be an at most uniquely complemented lattice with the $\mathcal{C}$-extension $\widehat{K}$.*

If S is isomorphic to a confined sublattice S_1 of $\widehat{K}$, then S_1 is contained in K.

Proof. This is now trivial from the previous result. By Theorem 15, S_1 is contained in K or F. But F is a $\{0,1\}$-free lattice, and a sublattice of a free lattice cannot be a simple lattice with more than two elements. $\square$

4. The lattice $\Gamma(A, \mathbf{G})$

This construction utilizes the lattice $\Gamma(A, \mathbf{G})$ of V. Koubek and J. Sichler [12], where A is an object of a certain category $\mathbf{D}(J)$ and $\mathbf{G}$ is a specific graph. We start with the description of $\mathbf{G}$, $\mathbf{D}(J)$, and the construction of $\Gamma(A, \mathbf{G})$.

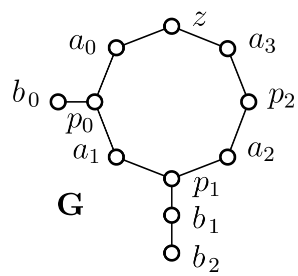

Figure 1a

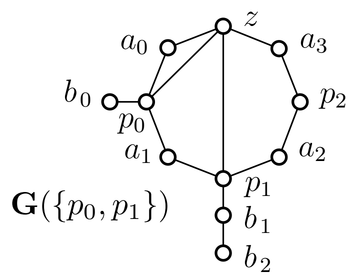

Figure 1b

4.1. **The graph G.** Let n be a natural number and let $\mathbf{G} = \langle V, E \rangle$ be the graph consisting of a cycle

$$\{z, a_0, p_0, a_1, p_1, \dots, a_{n-1}, p_{n-1}, a_n\}$$

and three more elements $\{b_0, b_1, b_2\}$, along with the additional edges

$$\{p_0, b_0\},\ \{p_1, b_1\},\ \{b_1, b_2\}.$$

See Figure 1a for the graph $\mathbf{G}$ with $n = 3$.

$\mathbf{G} = \langle V, E \rangle$ is a finite automorphism-free connected graph, so $\operatorname{Ind}\mathbf{G}$ is a finite simple lattice by Lemma 3. Let $J = \{p_0, p_1, \dots, p_{n-1}\} \subseteq V$. Then $\{z\} \cup J$ is an independent set. For any nonvoid $J_1 \subseteq J$, let $E(J_1) = \{\, \{z, p_j\} \mid j \in J_1 \,\}$ and $\mathbf{G}(J_1) = \langle V, E \cup E(J_1) \rangle$; see Figure 1b for the graph $\mathbf{G}(\{p_0, p_1\})$ with $n = 3$. Then $\mathbf{G}(J_1)$ is finite, automorphism-free, and connected, so that the lattice $\operatorname{Ind}\mathbf{G}(J_1)$ is simple by Lemma 3.

Let $\mathbf{H}$ be the full subgraph of $\mathbf{G}$ on the vertex set $V - \{z\}$. Since the independent sets of $\mathbf{H}$ are exactly the independent sets of $\mathbf{G}$ contained in $V - \{z\}$, the lattice $\operatorname{Ind}\mathbf{H}$ is a $\{0,1\}$-sublattice of $\operatorname{Ind}\mathbf{G}$. Moreover, $J \subseteq V - \{z\}$ implies that the lattice $\operatorname{Ind}\mathbf{H}$ is also a $\{0,1\}$-sublattice of $\operatorname{Ind}\mathbf{G}(J_1)$, for each nonvoid $J_1 \subseteq J$.

4.2. **The category $\mathbf{D}(J)$.** We now define the category $\mathbf{D}(J)$. Let J be a finite set. The *objects* of $\mathbf{D}(J)$ are the pairs $\langle D, \mathcal{P} \rangle$, where D is a distributive lattice with a zero, z, and $\mathcal{P}$ is a collection $\{\, P_j \mid j \in J \,\}$ of prime filters of D. The *morphisms* of $\mathbf{D}(J)$:

$$\varphi\colon \langle D, \mathcal{P} \rangle \to \langle D^\dagger, \mathcal{P}^\dagger \rangle$$

(where $\mathcal{P}^\dagger = \{\, P_j^\dagger \mid j \in J \,\}$) are the $\{0\}$-lattice homomorphisms $\varphi\colon D \to D^\dagger$ that satisfy $(P_j^\dagger)\varphi^{-1} = P_j$, for all $j \in J$.

Let us call an object $\langle D, \mathcal{P} \rangle$ of $\mathbf{D}(J)$ *sharp*, if D has a unit and every endomorphism of $\langle D, \mathcal{P} \rangle$ is a $\{0,1\}$-endomorphism of D.

The following statement is from V. Koubek and J. Sichler [12] (see also M.E. Adams, V. Koubek, and J. Sichler [2]):

Theorem 17. *For a sufficiently large finite set J, the category $\mathbf{D}(J)$ is finite-to-finite universal. In particular, every (finite) monoid $\mathcal{M}$ is isomorphic to the endomorphism monoid of some (finite) object in $\mathbf{D}(J)$; in fact, of a sharp object in $\mathbf{D}(J)$.*

4.3. **The construction $\Gamma(A, \mathbf{G})$.** Now for any sharp object $A = \langle D, \mathcal{P} \rangle$ of the category $\mathbf{D}(J)$, we construct a lattice $\Gamma(A, \mathbf{G})$, as in V. Koubek and J. Sichler [12], using the graph $\mathbf{G}$ of Section 4.1 with $n = |J|$. (The construction in V. Koubek and J. Sichler [12] is for a large collection of graphs, but we need it here only for our graph $\mathbf{G}$.)

Let z and u be zero and unit of the distributive lattice D, respectively. We extend D to a distributive lattice D^+ by adding a new zero, 0_{D^+}, and a new unit, 1_{D^+}. In the direct product

$$D^+ \times \operatorname{Ind} \mathbf{H}$$

(the graph $\mathbf{H}$ was introduced in Section 4.1), let R be the order filter generated by the set

$$\begin{aligned} R_0 = &\bigcup (\, \{\, \langle d, \{p_j\} \rangle \mid d \in P_j \,\} \mid j \in J \,) \\ &\cup \{\, \langle z, \{v\} \rangle \mid \{z, v\} \in E \,\} \cup \{ \langle 0_{D^+}, 1_{\mathbf{H}} \rangle, \langle 1_{D^+}, 0_{\mathbf{H}} \rangle \}. \end{aligned}$$

We form the poset

$$\Gamma(A, \mathbf{G}) = ((D^+ \times \operatorname{Ind} \mathbf{H}) - R) \cup \{1_\Gamma\},$$

where 1_Γ is the greatest element. Since R is an order filter, the poset $\Gamma(A, \mathbf{G})$ is a $\{0,1\}$-lattice. Let 0_Γ denote the zero of $\Gamma(A, \mathbf{G})$. Note that the interval $[\langle z, \varnothing \rangle, \langle u, \varnothing \rangle]$ of $\Gamma(A, \mathbf{G})$ is isomorphic to D.

The following statement is a useful rewrite of the definition of $\Gamma(A, \mathbf{G})$:

Lemma 18. *Each $g \in \Gamma(A, \mathbf{G}) - \{1_\Gamma\}$ is a pair $g = \langle d, I \rangle$, where $d \in D^+ - \{1_{D^+}\}$ and $I \subseteq V - \{z\}$ is an independent set of $\mathbf{H}$ such that either $d = 0_{D^+}$ or $d \in D$ and*

(i) *$\{z\} \cup I$ is an independent set of $\mathbf{G}$;*
(ii) *$p_j \notin I$, for $d \in P_j$, $j \in J$.*

The following result of V. Koubek and J. Sichler [12] describes the $\{0,1\}$-endomorphisms of $\Gamma(A, \mathbf{G})$.

Let φ be an endomorphism of A. Define a map $\Gamma(\varphi)$ of $\Gamma(A, \mathbf{G})$ into itself as follows:

$$g\Gamma(\varphi) = \begin{cases} g, & \text{for } g = 1_\Gamma \text{ and for } g = \langle 0_{D^+}, I \rangle; \\ \langle d\varphi, I \rangle, & \text{for } g = \langle d, I \rangle,\ d \in D. \end{cases}$$

Theorem 19. *The map $\varphi \mapsto \Gamma(\varphi)$ is an isomorphism between the endomorphism monoid of A and the $\{0,1\}$-endomorphism monoid of $\Gamma(A, \mathbf{G})$. Moreover, every endomorphism of $\Gamma(A, \mathbf{G})$ is either a $\{0,1\}$-endomorphism or a constant.*

We shall need some information about atoms and dual atoms of $\Gamma(A, \mathbf{G})$.

Lemma 20. *Every atom of the lattice $\Gamma(A, \mathbf{G})$ is fixed by all $\{0,1\}$-endomorphisms of $\Gamma(A, \mathbf{G})$. Let $I \cup \{z\}$ be a maximal independent set of $\mathbf{G}(J)$. Then $\langle u, I \rangle$ is a dual atom of $\Gamma(A, \mathbf{G})$ fixed by all $\{0,1\}$-endomorphisms of $\Gamma(A, \mathbf{G})$.*

Proof. This is obvious from the description of $\Gamma(\varphi)$ given above. □

5. Simple sublattices of $\Gamma(A, \mathbf{G})$

In this section, as a result of the special properties of the graph $\mathbf{G}$, we prove that every element of $\Gamma(A, \mathbf{G})$ is contained in a simple sublattice with more than two elements and not isomorphic to $\mathfrak{M}_3$.

If I is an independent set of $\mathbf{G}$, then either $z \notin I$ and I is independent in $\mathbf{H}$, or $z \in I$ and $I - \{z\}$ is independent in $\mathbf{H}$. For any $I \in \operatorname{Ind}\mathbf{G} - \{1_{\mathbf{G}}\}$, set

$$I\chi = \begin{cases} \langle 0_{D^+}, I\rangle, & \text{if } z \notin I; \\ \langle z, I - \{z\}\rangle, & \text{if } z \in I; \end{cases}$$

and set $1_{\mathbf{G}}\chi = 1_\Gamma$.

Lemma 21. *χ is an embedding of* $\operatorname{Ind}\mathbf{G}$ *into* $\Gamma(A, \mathbf{G})$.

Proof. Since $z \notin P_j$, for all $j \in J$, it follows that $I\chi < 1_\Gamma$ in $\Gamma(A, \mathbf{G})$ if and only if $I < 1_{\mathbf{G}}$ in $\operatorname{Ind}\mathbf{G}$. Therefore,

$$\chi\colon \operatorname{Ind}\mathbf{G} \to \Gamma(A, \mathbf{G})$$

is a well-defined mapping, and $\varnothing\chi = \langle 0_{D^+}, \varnothing\rangle$. It is clear that $I_1 < I_2$ in $\operatorname{Ind}\mathbf{G}$ if and only if $I_1\chi < I_2\chi$ in $\Gamma(A, \mathbf{G})$. Thus, in particular, χ is one-to-one and preserves the bounds.

To show that χ is a homomorphism, choose I_1, $I_2 \in \operatorname{Ind}\mathbf{G} - \{1_{\mathbf{G}}\}$. Assume that $I_1\chi \vee I_2\chi < 1_\Gamma$ in $\Gamma(A, \mathbf{G})$. We distinguish three cases.

Case 1. $z \notin I_1 \cup I_2$. Then $1_\Gamma > I_1\chi \vee I_2\chi = \langle 0_{D^+}, I_1\rangle \vee \langle 0_{D^+}, I_2\rangle$, and hence $I_1 \vee I_2 < 1_{\mathbf{G}}$ in $\operatorname{Ind}\mathbf{G}$. But then $I_1 \vee I_2 = I_1 \cup I_2$, and so

$$(I_1 \vee I_2)\chi = (I_1 \cup I_2)\chi = \langle 0_{D^+}, I_1 \cup I_2\rangle = I_1\chi \vee I_2\chi.$$

Case 2. $z \in I_1 - I_2$. Then $1_\Gamma > I_1\chi \vee I_2\chi = \langle z, I_1 - \{z\}\rangle \vee \langle 0_{D^+}, I_2\rangle$ in $\Gamma(A, \mathbf{G})$. In particular, $I = (I_1 - \{z\}) \vee I_2 < 1_{\mathbf{G}}$ in $\operatorname{Ind}\mathbf{G}$, and hence $I = (I_1 - \{z\}) \vee I_2 = (I_1 - \{z\}) \cup I_2 = (I_1 \cup I_2) - \{z\}$, and $I \cup \{z\} = I_1 \cup I_2$ is independent in $\mathbf{G}$. But then $I \cup \{z\} = I_1 \vee I_2$ in $\operatorname{Ind}\mathbf{G}$, and so $(I_1 \vee I_2)\chi = \langle z, I\rangle = I_1\chi \vee I_2\chi$.

Case 3. $z \in I_1 \cap I_2$. This is similar to Case 1.

This proves that $I_1\chi \vee I_2\chi < 1_\Gamma$ implies that $I_1\chi \vee I_2\chi = (I_1 \vee I_2)\chi$. Since χ preserves order, if $I_1\chi \vee I_2\chi = 1_\Gamma$, then $(I_1 \vee I_2)\chi = 1_\Gamma$. Therefore, χ preserves joins.

To show that χ preserves meets is much easier. Let I_1, $I_2 \in \operatorname{Ind}\mathbf{G} - \{1_{\mathbf{G}}\}$. If $z \in I_1 \cap I_2$, then $I_1\chi \wedge I_2\chi = \langle z, I_1 - \{z\}\rangle \wedge \langle z, I_2 - \{z\}\rangle = \langle z, (I_1 \cap I_2) - \{z\}\rangle = (I_1 \wedge I_2)\chi$, and the other cases follow similarly. ☐

This shows that all elements of $\Gamma(A, \mathbf{G})$ other than $\langle d, I\rangle$ with $d \in D - \{z\}$ belong to the simple $\{0,1\}$-sublattice $(\operatorname{Ind}\mathbf{G})\chi$ of $\Gamma(A, \mathbf{G})$.

Lemma 22. *All elements of $\Gamma(A, \mathbf{G})$ of the form $\langle d, I\rangle$ with $d \in D$ and $d > z$ belong to a simple $\{0,1\}$-sublattice isomorphic to* $\operatorname{Ind}\mathbf{G}(J_d)$*, where $J_d = \{\, p_j \mid d \in P_j \,\}$.*

Proof. Observe that J_d is nonvoid, so $\mathbf{G}(J_d)$ is defined. We want to construct a $\{0,1\}$-homomorphism $\chi\colon \operatorname{Ind}\mathbf{G}(J_d) \to \Gamma(A, \mathbf{G})$ so that the given $\langle d, I\rangle$ is in $\operatorname{Ind}\mathbf{G}(J_d)\chi$.

We set $1_{\mathbf{G}(J_d)}\chi = 1_\Gamma$ and, for any $I \in \operatorname{Ind}\mathbf{G}(J_d) - \{1_{\mathbf{G}(J_d)}\}$, define

$$I\chi = \begin{cases} \langle 0_{D^+}, I\rangle, & \text{if } z \notin I; \\ \langle d, I - \{z\}\rangle, & \text{if } z \in I. \end{cases}$$

We need to check that χ is well-defined. Any $I < 1_{\mathbf{G}(J_d)}$ not containing z is an independent set of $\mathbf{H}$, and hence $I\chi = \langle 0_{D^+}, I\rangle \in \Gamma(A, \mathbf{G}) - \{1_\Gamma\}$. If $z \in I$, then $I - \{z\}$ is an independent set of $\mathbf{H}$ and $(I - \{z\}) \cup \{z\} = I$ is independent in $\mathbf{G}$. Thus (i) of Lemma 18 holds. Also, $I \cap J_d = \varnothing$ holds because $z \in I$ and $\{z, j'\}$ is an edge of $\mathbf{G}(J_d)$, for every $j' \in J_d = \{\, j \in J \mid d \in P_j \,\}$; thus (ii) of Lemma 18 also holds, and hence $I\chi < 1_\Gamma$, for all $I < 1_{\mathbf{G}(J_d)}$. Clearly, $\varnothing\chi = \langle 0_{D^+}, \varnothing\rangle$.

Let I_1, $I_2 \in \operatorname{Ind} \mathbf{G}(J_d) - \{1_{\mathbf{G}(J_d)}\}$. It is easy to check that $I_1 < I_2$ if and only $I_1\chi < I_2\chi$. It follows that $q\chi \vee r\chi \leq (q \vee r)\chi$, for all q, $r \in \operatorname{Ind} \mathbf{G}(J_d)$. In particular, if $q\chi \vee r\chi = 1_\Gamma$, then $(q \vee r)\chi = 1_\Gamma$. To show that χ preserves joins, we thus need only check the case when I_1, $I_2 \in \operatorname{Ind} \mathbf{G}(J_d) - \{1_{\mathbf{G}(J_d)}\}$ and $I_1\chi \vee I_2\chi < 1_\Gamma$.

We proceed as before, distinguishing three cases.

Case 1. $z \notin I_1 \cup I_2$. Then $I_1\chi \vee I_2\chi = \langle 0_{D^+}, I_1\rangle \vee \langle 0_{D^+}, I_2\rangle < 1_\Gamma$ in $\Gamma(A, \mathbf{G})$, and hence $I_1 \vee I_2 < 1_{\mathbf{G}}$ in $\operatorname{Ind} \mathbf{G}$. But then $I_1 \vee I_2 = I_1 \cup I_2$ in $\operatorname{Ind} \mathbf{G}$, so that $I_1 \cup I_2$ is an independent set of $\mathbf{G}$ as well, and $(I_1 \vee I_2)\chi = (I_1 \cup I_2)\chi = \langle 0_{D^+}, I_1 \cup I_2\rangle = I_1\chi \vee I_2\chi$.

Case 2. $z \in I_1 - I_2$. Then $1_\Gamma > I_1\chi \vee I_2\chi = \langle d, I_1 - \{z\}\rangle \vee \langle 0_{D^+}, I_2\rangle$ in $\Gamma(A, \mathbf{G})$. In particular, $I = (I_1 - \{z\}) \vee I_2 < 1_{\mathbf{G}}$ in $\operatorname{Ind} \mathbf{G}$, and hence $I = (I_1 - \{z\}) \vee I_2 = (I_1 - \{z\}) \cup I_2 = (I_1 \cup I_2) - \{z\}$, so that $I \cup \{z\} = I_1 \cup I_2$ is independent in $\mathbf{G}$, see (i). Therefore, $I_1\chi \vee I_2\chi = \langle d, I\rangle$. For any $j' \in J_d$, we have that $d \in P(j')$, and hence $j' \notin I$, by (ii). This means that $I \cap J_d = \varnothing$, and the definition of $\mathbf{G}(J_d)$ implies that $I \cup \{z\} = I_1 \cup I_2$ is an independent set of $\mathbf{G}(J_d)$ as well. From the definition of χ, we then get $(I_1 \vee I_2)\chi = (I_1 \cup I_2)\chi = \langle d, (I_1 \cup I_2) - \{z\}\rangle = \langle d, I\rangle$, and $(I_1 \vee I_2)\chi = I_1\chi \vee I_2\chi$ follows.

Case 3. $z \in I_1 \cap I_2$. Then $1_\Gamma > I_1\chi \vee I_2\chi = \langle d, I_1 - \{z\}\rangle \vee \langle d, I_2 - \{z\}\rangle$ in $\Gamma(A, \mathbf{G})$, so that $I = (I_1 - \{z\}) \vee (I_2 - \{z\}) < 1_{\mathbf{G}}$ in $\operatorname{Ind} \mathbf{G}$, and hence $I = (I_1 \cup I_2) - \{z\}$ in $\operatorname{Ind} \mathbf{G}$. Thus $I \cup \{z\} = I_1 \cup I_2$ is independent in $\mathbf{G}$ and $I_1\chi \vee I_2\chi = \langle d, I\rangle$. For any $j' \in J_d$, we have that $d \in P'_j$, and hence $j' \notin I$. This means that $I \cap J_d = \varnothing$, and this implies that $I \cup \{z\} = I_1 \cup I_2$ is an independent set of $\mathbf{G}(J_d)$. From the definition of χ, we then get

$$(I_1 \vee I_2)\chi = (I_1 \cup I_2)\chi = \langle d, (I_1 \cup I_2) - \{z\}\rangle = \langle d, I\rangle,$$

and $(I_1 \vee I_2)\chi = I_1\chi \vee I_2\chi$ follows.

This proves that χ preserves joins, and it is clear how to show that χ preserves meets. □

Summarizing the results of this section, we obtain

Theorem 23. *The lattice* $\Gamma(A, \mathbf{G})$ *is a union of simple* $\{0,1\}$*-sublattices isomorphic to one of the lattices* $\operatorname{Ind} \mathbf{G}$ *and* $\operatorname{Ind} \mathbf{G}(J_1)$*, where* $J_1 \subseteq J$.

6. Proof of the Main Theorem

For a monoid $\mathcal{M}$, let A be a sharp object in the category $\mathbf{D}(J)$ (see Section 4.2) whose endomorphism monoid is isomorphic to $\mathcal{M}$. Let $\mathbf{G}$ be the graph introduced in Section 4.1, and let $\Gamma(A, \mathbf{G})$ be the lattice constructed in Section 4.3.

To start our construction of the lattice L of the Main Theorem, we need an at most uniquely complemented lattice L_0 with $\operatorname{End}_{\{0,1\}} L_0 \cong \mathcal{M}$ and with the property that every element x of $L_0 - \{0_{L_0}, 1_{L_0}\}$ is contained in a confined simple sublattice S of L_0 with some special properties.

By Lemma 20, there is a dual atom q of $\Gamma(A, \mathbf{G})$ such that every $\{0,1\}$-endomorphism of $\Gamma(A, \mathbf{G})$ keeps q fixed.

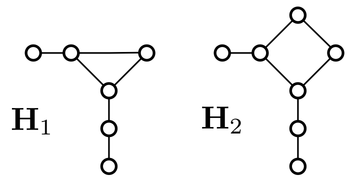

Figure 2

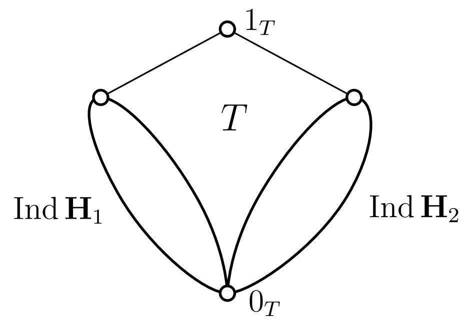

Figure 3

Let p be any atom of $\Gamma(A, \mathbf{G})$ satisfying $p \leq q$; we remarked at the end of Section 4.3 that every $\{0,1\}$-endomorphism of $\Gamma(A, \mathbf{G})$ keeps p fixed.

Now let $\mathbf{H}_1$ and $\mathbf{H}_2$ be the graphs of Figure 2. From $\operatorname{Ind} \mathbf{H}_1$ and $\operatorname{Ind} \mathbf{H}_2$, we construct a lattice T as follows (see Figure 3):

$$T = \operatorname{Ind} \mathbf{H}_1 \cup \operatorname{Ind} \mathbf{H}_2 \cup \{1_T\},$$

where T is a disjoint union of the three sets except that $0_{\operatorname{Ind} \mathbf{H}_1} = 0_{\operatorname{Ind} \mathbf{H}_2}$. We partially order T as follows: $\operatorname{Ind} \mathbf{H}_1$ and $\operatorname{Ind} \mathbf{H}_2$ are sublattices of T; 1_T is the unit element; for $x \in \operatorname{Ind} \mathbf{H}_1 - \{0_{\operatorname{Ind} \mathbf{H}_1}\}$ and $y \in \operatorname{Ind} \mathbf{H}_2 - \{0_{\operatorname{Ind} \mathbf{H}_2}\}$, let $x \wedge y = 0_T$ and $x \vee y = 1_T$.

By Lemmas 3 and 4, the lattices $\operatorname{Ind} \mathbf{H}_1$ and $\operatorname{Ind} \mathbf{H}_2$ are simple complemented lattices with no nontrivial $\{0,1\}$-endomorphisms. Let a be any atom of T.

Lemma 24. *T is a simple complemented lattice with no nontrivial $\{0,1\}$-endomorphism. Every non-unit element x of T is contained in a simple sublattice U such that $1_T \notin U$. If a congruence Θ of T collapses U, then it collapses all of T.*

Proof. Indeed, we can always take $U = \operatorname{Ind} \mathbf{H}_1$ or $U = \operatorname{Ind} \mathbf{H}_2$. $\square$

We define the lattice B as the dual of T. The image of the atom a of A in B will be denoted by b, a dual atom of B.

We construct L_0 by gluing $\Gamma(A, \mathbf{G})$ to B and T. First, we glue the two-element filter $\{b, 1_B\}$ of B to the two-element ideal $\{0_\Gamma, p\}$ of $\Gamma(A, \mathbf{G})$. Then we glue the two-element filter $\{q, 1_\Gamma\}$ of the resulting lattice to the two-element ideal $\{0_T, a\}$ of T, to obtain L_0; see Figure 4. Thus $0_{L_0} = 0_B$ is the zero of L_0 and $1_{L_0} = 1_T$ is the unit of L_0.

We regard B, $\Gamma(A, \mathbf{G})$, and T as sublattices of L_0; in L_0, we have $b = 0_\Gamma$, $1_B = p$, $q = 0_T$, and $1_\Gamma = a$.

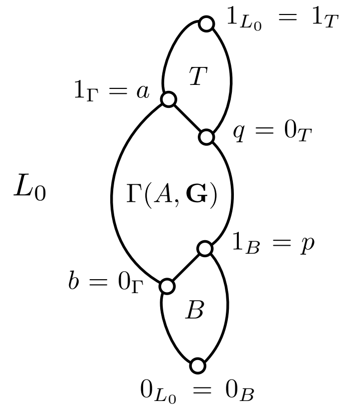

Figure 4

We can easily describe the $\{0,1\}$-endomorphisms of L_0. Let φ be a $\{0,1\}$-endomorphism of L_0, and let $\overline{\varphi}$ be the trivial extension of φ to L_0, that is,

$$g\overline{\varphi} = \begin{cases} g\varphi, & \text{for } g \in \Gamma(A, \mathbf{G}); \\ g, & \text{for } g \in B \cup T. \end{cases}$$

Note that this definition makes sense since φ fixes p and q.

Lemma 25. *The map $\varphi \mapsto \overline{\varphi}$ is an isomorphism between the endomorphism monoids* $\operatorname{End}_{\{0,1\}} \Gamma(A, \mathbf{G})$ *and* $\operatorname{End}_{\{0,1\}} L_0$. *Therefore,*

$$\operatorname{End}_{\{0,1\}} L_0 \cong \mathcal{M}.$$

Proof. Let ψ be a $\{0,1\}$-endomorphism of L_0. The lattice $B \cup \operatorname{Ind} \mathbf{G} \cup T$ (with $\operatorname{Ind} \mathbf{G} \subseteq \Gamma(A, \mathbf{G})$) is simple, so that ψ restricted to B is one-to-one. In L_0, the largest element e such that $[0_{L_0}, e]$ is complemented is $e = p$ $(= 1_B)$. Hence $p = 1_B$ is fixed by ψ; moreover, ψ restricted to B is the identity map on B. Similarly, $0_T = q$ is fixed by ψ; moreover, ψ restricted to T is the identity map on T. In particular, $0_\Gamma \psi = 0_\Gamma$ and $1_\Gamma \psi = 1_\Gamma$.

Since 0_Γ and 1_Γ are fixed by ψ, it follows that ψ restricted to $\Gamma(A, \mathbf{G})$ is a $\{0,1\}$-endomorphism φ of $\Gamma(A, \mathbf{G})$. ☐

We need one more trivial statement:

Lemma 26. $\{0_{L_0}, 1_{L_0}\}$ *is the only complemented pair in* L_0.

Proof. Indeed, let $\{x, y\}$ be a complemented pair in L_0. If $x \in B - \Gamma(A, \mathbf{G})$, then $x \wedge y = 0_{L_0} = 0_B$ implies that $y \in B \cup \Gamma(A, \mathbf{G})$ (since $p \leq q$), and so $x \vee y = 1_{L_0}$ is not possible. If $x \in \Gamma(A, \mathbf{G}) - B$ or if $x \in T - \Gamma(A, \mathbf{G})$, then we argue similarly. Finally, $x \in \{b, p\}$ or $x \in \{q, a\}$ is clearly not possible. ☐

By Lemma 26, L_0 is at most uniquely complemented, so we can define L_1 as the $\mathcal{C}$-extension $\widehat{L_0}$ of L_0. By Theorem 14, every $\{0,1\}$-endomorphism of L_0 has a unique extension to a $\{0,1\}$-endomorphism of L_1.

Lemma 27. *Every $\{0,1\}$-endomorphism of L_1 is the unique extension of a $\{0,1\}$-endomorphism of L_0 to L_1. Therefore,*

$$\operatorname{End}_{\{0,1\}} L_1 \cong \mathcal{M}.$$

Proof. Let ψ be a $\{0,1\}$-endomorphism of L_1. We have to prove that $x\psi \in L_0$, for every $x \in L_0$. The lemma then follows from Lemma 25. There are three cases to consider.

Case 1. $x \in T$. The statement is obvious for $x = 1_T$ $(= 1_{L_0})$, so let $x < 1_T$. Then by Lemma 24, there is a sublattice U of T such that $1_T \notin U$, U is a simple lattice, and if a congruence Θ collapses U, then it collapses all of T.

Consider the sublattice $U\psi$ of L_1. Observe that $1_{L_1} \notin U\psi$. Indeed, if $1_{L_1} \in U\psi$, then $1_U\psi = 1_{L_0}\psi$, where 1_U is the unit element of U. Since $1_U < 1_{L_0}$ and T is simple, this implies that all of T is collapsed by ψ to one element. In particular, $q\psi = 1_\Gamma\psi$. Now use Theorem 23, and obtain a simple $\{0,1\}$-sublattice V of $\Gamma(A, \mathbf{G})$ containing q. Then $V\psi$ is a singleton; in particular, $0_\Gamma\psi = p\psi$. Proceeding now by the dual of Lemma 24, we obtain that $1_B\psi = 0_B\psi$, and so $0_{L_1}\psi = 1_{L_1}\psi$, contradicting the fact that ψ is a $\{0,1\}$-endomorphism of L_1.

We conclude that $1_{L_1} \notin U\psi$. The argument of the previous paragraph can be repeated to show that ψ is one-to-one on U. Hence, $U\psi$ is a simple lattice in L_1; it does not contain 1_{L_1}; and it is not isomorphic to $\mathfrak{M}_3$ (because it is isomorphic to $\operatorname{Ind}\mathbf{H}_1$ or $\operatorname{Ind}\mathbf{H}_2$). So we can apply Theorem 16 to conclude that $U \subseteq L_0$, and in particular, $x\psi \in L_0$.

Case 2. $x \in \Gamma(A, \mathbf{G})$. By Case 1, we can assume that $x < 1_\Gamma$. Now we proceed by applying Theorem 23 as in Case 1, and then utilize the fact that B and T are simple lattices.

Case 3. $x \in B$. This is the dual of Case 1. ☐

So now we see that L_1 is at most uniquely complemented, and $\operatorname{End}_{\{0,1\}} L_1 \cong \mathcal{M}$ by natural extension from L_0. Moreover, every element of L_0 has a complement in L_1. Let us now repeat this construction for L_1 to obtain the $\mathcal{C}$-extension $L_2 = \widehat{L_1}$ of L_1. Now we can restate Lemma 27 for L_0 and L_2:

Lemma 28. *Every $\{0,1\}$-endomorphism of L_2 is the unique extension of a $\{0,1\}$-endomorphism of L_0 to L_2. Therefore,*

$$\operatorname{End}_{\{0,1\}} L_2 \cong \mathcal{M}.$$

Proof. By Theorem 14, every $\{0,1\}$-endomorphism φ of L_1 has a unique extension to a $\{0,1\}$-endomorphism of L_2. Let ψ be a $\{0,1\}$-endomorphism of L_2. By Lemma 27, it is sufficient to show that that $x\psi \in L_0$ for every $x \in L_0$. So let $x \in L_0 - \{0,1\}$ and, as in the proof of Lemma 27, choose a simple finite sublattice U of L_0 such that $x \in U$, 0_{L_0}, $1_{L_0} \notin U$, if a congruence Θ collapses U, then it collapses all of L_0, and U is not isomorphic to $\mathfrak{M}_3$. It follows then that $U\psi$ is isomorphic to U. So we can apply Theorem 16 to conclude that $U \subseteq L_1$. But by Lemma 27, $U \subseteq L_1$ implies that $U \subseteq L_0$. Hence $x \in L_0$, which was to be proved. ☐

Note that Lemma 28 required a proof because L_1 does not have a crucial property of L_0. Now we see that L_0 in L_2 has all the important properties that L_0 has in L_1. So we repeat the $\mathcal{C}$-extension construction ω times, to obtain the lattices L_i, for $i < \omega$, and we obtain the lattice L of the Main Theorem as the union of the L_i, for $i < \omega$. A trivial induction on i shows that every $\{0,1\}$-endomorphism of L_i is the unique extension of a $\{0,1\}$-endomorphism of L_0 to L_i. The induction step repeats the argument of Lemma 28. This trivially implies that every $\{0,1\}$-endomorphism of L is the unique extension of a $\{0,1\}$-endomorphism of L_0 to L. Therefore,

$$\operatorname{End}_{\{0,1\}} L \cong \mathcal{M}.$$

Moreover, every $x \in L_i$ has a complement in L_{i+1} and each L_i is at most uniquely complemented; hence L is uniquely complemented. This concludes the proof of the Main Theorem.

7. Proof of Theorem 2

We need a definition:

Definition 29. Let φ be a $\{0,1\}$-endomorphism of a bounded lattice N. We shall say that φ is $\{0,1\}$-*separating*, if $0\varphi^{-1} = \{0\}$ and $1\varphi^{-1} = \{1\}$.

Our construction for Theorem 2 starts with a result of V. Koubek and J. Sichler [12]:

Theorem 30. *There is a class* $\mathbf{N}$ *of bounded lattices with the following properties:*

1. $\mathbf{N}$ *is contained in a finitely generated lattice variety.*
2. *If* N, $N' \in \mathbf{N}$, *then any nonconstant homomorphism* $\varphi\colon N \to N'$ *is a* $\{0,1\}$-*separating* $\{0,1\}$-*homomorphism.*
3. *The nonconstant homomorphisms among members of* $\mathbf{N}$ *form a finite-to-finite universal category.*

According to the last clause, all monoids can be represented as $\{0,1\}$-endomorphism monoids of members of $\mathbf{N}$, and finite ones by finite members.

Let $\mathbf{H}_1 = \langle V, E\rangle$ be the graph of Figure 2. Then $\operatorname{Ind}\mathbf{H}_1$ is a simple automorphism-free complemented lattice. We note that $\mathbf{H}_1$ has the following property:

(P) for every $v \in V$, there exist distinct v_1, $v_2 \in V$ so that $v \neq v_1$, $v \neq v_2$, and $\{v, v_1\}$, $\{v, v_2\} \notin E$.

Choose distinct vertices x_0 and x_1 of $\mathbf{H}_1$ so that $\{x_0, x_1\} \notin E$. Select $N \in \mathbf{N}$ representing the monoid $\mathcal{M}$. From $\mathbf{H}_1$, $\{x_0, x_1\}$, and N, we construct a poset $(\operatorname{Ind}\mathbf{H}_1)[N]$ by replacing the prime interval $[\{x_0\}, \{x_0, x_1\}]$ of $\operatorname{Ind}\mathbf{H}_1$ by a copy of N, so that $\{x_0\} = 0_N$ and $\{x_0, x_1\} = 1_N$, see Figure 5. To be more precise, the partial ordering on $(\operatorname{Ind}\mathbf{H}_1)[N]$ is defined as follows:

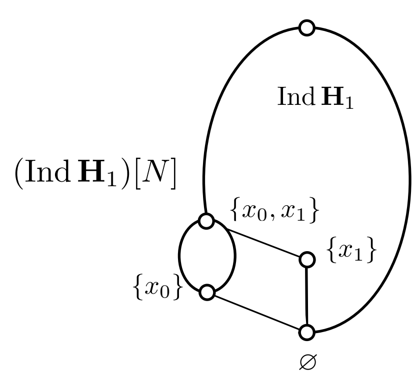

Figure 5

$x \leq y$ in $(\operatorname{Ind} \mathbf{H}_1)[N]$ iff
$x, y \in \operatorname{Ind} \mathbf{H}_1$ and $x \leq y$ in $\operatorname{Ind} \mathbf{H}_1$;
$x, y \in N$ and $x \leq y$ in N;
$x \in N$ and $y \in \operatorname{Ind} \mathbf{H}_1$ with $\{x_0, x_1\} \leq y$ in $(\operatorname{Ind} \mathbf{H}_1)[N]$;
$x \in \operatorname{Ind} \mathbf{H}_1$ and $y \in N$ and $x \in \{\varnothing, \{x_0\}\}$.

It is easy to verify that $(\operatorname{Ind} \mathbf{H}_1)[N]$ is a lattice. For any $n \in N - \{0_N, 1_N\}$ and for $t \in \operatorname{Ind} \mathbf{H}_1$, we have

$$\begin{aligned} n \vee t &= \{x_0, x_1\} \vee t, && \text{for } t \nleq \{x_0\}; \\ n \wedge t &= \{x_0\} \wedge t, && \text{for } \{x_0, x_1\} \nleq t. \end{aligned}$$

Thus in N and in $\operatorname{Ind} \mathbf{H}_1$, all meets and joins remain the same. All other meets and joins are determined by comparability.

Let φ be a $\{0,1\}$-endomorphism of N. A *trivial extension* $\overline{\varphi}$ of φ to $(\operatorname{Ind} \mathbf{H}_1)[N]$ is defined as follows:

$$\begin{aligned} n\overline{\varphi} &= n\varphi, && \text{for } n \in N; \\ x\overline{\varphi} &= x, && \text{for } x \in \operatorname{Ind} \mathbf{H}_1. \end{aligned}$$

Lemma 31. *The lattice* $(\operatorname{Ind} \mathbf{H}_1)[N]$ *is complemented. The endomorphisms of* $(\operatorname{Ind} \mathbf{H}_1)[N]$ *are exactly the trivial extensions of* $\{0,1\}$*-endomorphisms of the lattice* N.

Proof. Since $\operatorname{Ind} \mathbf{H}_1$ is complemented, so is $(\operatorname{Ind} \mathbf{H}_1)[N]$.

Let φ be a $\{0,1\}$-endomorphism of N with the trivial extension $\overline{\varphi}$ to $(\operatorname{Ind} \mathbf{H}_1)[N]$. We claim that $\overline{\varphi} \in \operatorname{End}_{\{0,1\}}(\operatorname{Ind} \mathbf{H}_1)[N]$. To prove this, we need only consider meets and joins of elements $n \in N - \{0_N, 1_N\}$ and $t \in \operatorname{Ind} \mathbf{H}_1$. We want to prove that

$$(n \vee t)\overline{\varphi} = n\overline{\varphi} \vee t\overline{\varphi}.$$

For $t \nleq \{x_0\}$, we have that $n \vee t = \{x_0, x_1\} \vee t \in \operatorname{Ind} \mathbf{H}_1$ and hence $(n \vee t)\overline{\varphi} = \{x_0, x_1\} \vee t$, and also $n\overline{\varphi} \vee t\overline{\varphi} = n\varphi \vee t = \{x_0, x_1\} \vee t$, proving the formula. The argument for meets is similar but uses the fact that φ is $\{0,1\}$-separating.

Next, let $\gamma \in \operatorname{End}_{\{0,1\}}(\operatorname{Ind} \mathbf{H}_1)[N]$. We have to show that $\gamma = \overline{\varphi}$, for some $\varphi \in \operatorname{End}_{\{0,1\}} N$.

We claim that γ maps the sublattice $\operatorname{Ind} \mathbf{H}_1 \subseteq (\operatorname{Ind} \mathbf{H}_1)[N]$ into itself.

Suppose that $t \in \operatorname{Ind} \mathbf{H}_1$ is such that $t\gamma = n \in N - \{0_N, 1_N\}$. Then $\varnothing < t < 1_{\mathbf{G}}$ in $\operatorname{Ind} \mathbf{H}_1$, and hence $\{y\} \leq t$, for some $y \in V$. There are two cases to consider.

Case 1. $\{y\} < t$. Set $s = t - \{y\}$ and note that $\{y\} \vee s = t$ and $\{y\} \wedge s = \varnothing$ in $\operatorname{Ind} \mathbf{H}_1$. Since $\operatorname{Ind} \mathbf{H}_1$ is simple by Lemma 3, the four elements $\varnothing\gamma = \varnothing$, $\{y\}\gamma$, $s\gamma$, and $n = t\gamma$ are distinct. From $\{y\}\gamma \vee s\gamma = n \in N - \{0_N, 1_N\}$ and the description of joins in $(\operatorname{Ind} \mathbf{H}_1)[N]$, it then follows that $\{y\}\gamma$, $s\gamma \in N - \{0_N, 1_N\}$ and, consequently, $\varnothing = \varnothing\gamma \geq \{x_0\}$, a contradiction.

Case 2. $\{y\} = t$. $\mathbf{H}_1$ has property (P); therefore, there exist y_1, $y_2 \in V$, so that $\{y, y_1\}$, $\{y, y_2\} < 1_{\mathbf{G}}$ in $\operatorname{Ind} \mathbf{H}_1$. Observe that $\{y, y_1\} \wedge \{y, y_2\} = \{y\}$. This, however, leads to a contradiction because $\{y, y_i\}\gamma \notin N - \{0_N, 1_N\}$, for $i = 1$, 2, while $\{y\}\gamma = n \in N - \{0_N, 1_N\}$—recall that γ must be one-to-one on these three elements. This shows that

$$(N - \{0_N, 1_N\})\gamma^{-1} \cap \operatorname{Ind} \mathbf{H}_1 = \varnothing$$

and hence $(\operatorname{Ind} \mathbf{H}_1)\gamma \subseteq \operatorname{Ind} \mathbf{H}_1$, as claimed.

From this claim, it follows that γ is the identity on $\operatorname{Ind}\mathbf{H}_1$ and, consequently, $N\gamma \subseteq N$. Since the restriction φ of γ to N is not constant, it is a $\{0,1\}$-separating $\{0,1\}$-endomorphism of N and $\gamma = \overline{\varphi}$. ☐

So the lattice $L = (\operatorname{Ind}\mathbf{H}_1)[N]$ satisfies the requirements of Theorem 2.

References

1. M.E. Adams and J. Sichler, *Bounded endomorphisms of lattices of finite height*, Canad. J. Math. **24** (1977), 1254–1263. MR **56:**5374
2. M.E. Adams, V. Koubek, and J. Sichler, *Homomorphisms and endomorphisms of distributive lattices*, Houston J. Math. **11** (1985), 129–145. MR **86m:**06016
3. C.C. Chen and G. Grätzer, *On the construction of complemented lattices*, J. Algebra **11** (1969), 56–63. MR **38:**1038
4. R.P. Dilworth, *Lattices with unique complements*, Trans. Amer. Math. Soc. **57** (1945), 123–154. MR **7:**1b
5. ———, *The Dilworth theorems. Selected papers of Robert P. Dilworth*, Edited by Kenneth P. Bogart, Ralph Freese, and Joseph P.S. Kung. Contemporary Mathematicians. Birkhäuser Boston, Inc., Boston, MA, 1990. xxvi+465 pp. MR **92f:**01002
6. G. Grätzer, *A reduced free product of lattices*, Fund. Math. **73** (1971), 21–27. MR **46:**71010
7. ———, *Free products and reduced free products of lattices*, Proceedings of the University of Houston Lattice Theory Conference (Houston, Tex., 1973), pp. 539–563. Dept. Math., Univ. Houston, Houston, Tex. MR **53:**218
8. ———, *General Lattice Theory*, Pure and Applied Mathematics **75**, Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers), New York-London; Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, Band 52. Birkhäuser Verlag, Basel-Stuttgart; Akademie Verlag, Berlin, 1978. xiii+381 pp. MR **80c:**06001a,b
9. ———, *General Lattice Theory. Second Edition,* Birkhäuser Verlag, Basel, 1998. xix+663 pp. CMP 99:07
10. G. Grätzer and J. Sichler, *On the endomorphism semigroup (and category) of bounded lattices*, Pacific J. Math. **35** (1970), 639–647. MR **43:**3175
11. ———, *On the endomorphism monoid of complemented lattices*, AMS Abstract 97T-06-98.
12. V. Koubek and J. Sichler, *Universality of small lattice varieties*, Proc. Amer. Math. Soc. **91** (1984), 19–24. MR **85m:**06015
13. H. Lakser, *Simple sublattices of free products of lattices*, Abstract, Notices Amer. Math. Soc. **19** (1972), A 509.

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